Regular triangulations as lexicographic optimal chains

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Abstract

We introduce a total order on n-simplices in the n-Euclidean space for which the support of the lexicographic-minimal chain with the convex hull boundary as boundary constraint is precisely the n-dimensional Delaunay triangulation, or in a more general setting, the regular triangulation of a set of weighted points. This new characterization of regular and Delaunay triangulations is motivated by its possible generalization to submanifold triangulations as well as the recent development of polynomial-time triangulation algorithms taking advantage of this order.

1 Introduction

Algorithms computing the Delaunay triangulation and its variants [2] can be described as a set of combinatorial operations, evaluating geometric predicates and acting on algebraic quantities such as coordinates or equations. However, the Delaunay triangulation, or more generally regular triangulations, also have a variational formulation, introduced in [6] and further studied in [1, 5], characterizing them as solutions of a linear programming problem on triangulations. We show that this formulation can in fact be extended to the space of simplicial chains (Proposition 4.3).

The main part of this work consists in proving this variational formulation can be transposed in terms of lexicographic minimum (Theorem 3.1), for which polynomial time algorithms have recently been devised [8]. The proof of the theorem may be of interest in itself by offering an unusual point of view on regular triangulations.

One motivation for this work is that the same lexicographic order may be applied to the meshing of submanifold of Euclidean space. To be usable in this context, a such order must be isometry invariant, so that it can be evaluated from edge lengths only. It has been shown indeed that minimum chains in their homology class, for this lexicographic order, provide triangulations of well sampled (with respect to the reach) smooth 2-submanifolds of Euclidean n-space [7], akin to the tangential Delaunay complex [4]. In practice, this order enables efficient algorithms providing minimal solutions that inherit the optimality properties of 2-dimensional Delaunay triangulations and create pertinent and convincing meshes for surface reconstruction, in particular in the case of noisy point clouds with non uniform densities and outliers [8].

2 Conventions and notations

A k-simplex σ being a set of $k+1$ vertices in ambient space $\mathbb{R}^n$, we allow ourselves to use set theoretic operators on simplices. For example, $\tau \subset \sigma$ means that τ is a face of σ and $\sigma_1 \cup \sigma_2$ is the join of σ_1 and σ_2. $|\sigma|$ denotes the underlying space of the simplex σ, i.e. the convex hull of its vertices which, thanks to the generic condition 3.1, are affinely independent.

Homology coefficients are implicitly in $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, i.e. integers modulo 2 so that $1 = -1$. The vector space of k-chains over K is denoted $\boldsymbol{C_k}(K)$ in place of $\boldsymbol{C_k}(K, \mathbb{Z}_2)$. In this context we can allow ourselves to see chains as sets of simplices. For example, for $\Gamma \in \boldsymbol{C_k}(K)$ and σ is a

k-simplex in K, we write interchangeably $\sigma \in \Gamma$ and $\Gamma(\sigma) = 1$. Similarly, for $\Gamma_1, \Gamma_2 \in \boldsymbol{C_k}(K)$, we use interchangeably vector and set theoretic operators: $\Gamma_1 + \Gamma_2 = \Gamma_1 - \Gamma_2 = (\Gamma_1 \cup \Gamma_2) \setminus (\Gamma_1 \cap \Gamma_2)$.

For a chain Γ in $\boldsymbol{C_k}(K)$, we denote by $|\Gamma|$ the *support* of Γ, which is the sub-complex of K made of all k-simplices in Γ together with all their faces.

A total order $\leq$ on k-simplices induces a lexicographic order on k-chains with coefficients in $\mathbb{Z}_2$ as follows:

Definition 2.1 (Lexicographic Order on chains)**.** *Assume there is a total order $\leq$ on the set of k-simplices of K, defining the* max *on sets of k-simplices. For $\Gamma_1, \Gamma_2 \in \boldsymbol{C_k}(K)$:*

$$\Gamma_1 \sqsubseteq_{lex} \Gamma_2 \underset{def.}{\iff} \begin{cases} \Gamma_1 = \Gamma_2 \\ \text{or} \\ \max\{\sigma \in \Gamma_1 + \Gamma_2\} \in \Gamma_2 \end{cases}$$

3 Main result

We consider a set $\mathbf{P} = \{(P_1, \mu_1), \ldots, (P_N, \mu_N)\} \subset \mathbb{R}^n \times \mathbb{R}$ of weighted points in n-dimensional Euclidean space. A weighted point $(P, 0)$ is seen as a usual point $P \in \mathbb{R}^n$, while, when $\mu > 0$, it is associated to the sphere centered at P with radius $r = \sqrt{\mu}$. The *convex hull* $\mathcal{CH}(\mathbf{P})$ denotes the convex hull in $\mathbb{R}^n$of the set of points: $\mathcal{CH}(\mathbf{P}) = \mathcal{CH}(\{P_1, \ldots, P_N\})$.

The n-dimensional *full complex over* $\mathbf{P}$, denoted $K_{\mathbf{P}}$, is the simplicial complex made of all possible simplices up to dimension n with vertices in $\mathbf{P}$. The aim of this paper is to prove the following:

Theorem 3.1. *Let $\mathbf{P} = \{(P_1, \mu_1), \ldots, (P_N, \mu_N)\} \subset \mathbb{R}^n \times \mathbb{R}$, with $N \geq n+1$ be in general position and let $K_{\mathbf{P}}$ be the n-dimensional full complex over $\mathbf{P}$. Denote by $\beta_{\mathbf{P}} \in \boldsymbol{C_{n-1}}(K_{\mathbf{P}})$ the $(n-1)$-chain made of simplices belonging to the boundary of $\mathcal{CH}(\mathbf{P})$. If*

$$\Gamma_{\min} = \min_{\sqsubseteq_{lex}} \{\Gamma \in \boldsymbol{C_n}(K_{\mathbf{P}}), \partial\Gamma = \beta_{\mathbf{P}}\}$$

the simplicial complex $|\Gamma_{\min}|$ support of $\Gamma_{\min}$ is the regular triangulation of $\mathbf{P}$.

When all the weights are zero ($\forall i, \mu_i = 0$), the regular triangulation (Definition 4.1) is the Delaunay triangulation.

The relation $\sqsubseteq_{lex}$ among n-chains is the lexicographic order defined according to Definition 2.1, where the total order on n-simplices is given at the end of Section 3.3. The general position assumption is a generic condition formalized in Condition 3.2.

Obviously, replacing in Theorem 3.1 the full complex $K_{\mathbf{P}}$ by a complex containing the regular triangulation would again give this triangulation as a minimum.

3.1 Outline of the proof.

The main argument of the proof is given in Section 6.

Statement of Proposition 4.3 is the same as Theorem 3.1 except for the order along which the minimum is taken. In Proposition 4.3, the minimization is meant for the preorder (which is in fact generically an order) $\sqsubseteq_p$, induced by a weighted L^1 norm $\|\cdot\|_{(p)}$ on chains. All the proof consists then in showing that, while the two orders differ, they share the same minimum under the theorem constraints. The *bounding weight* of a simplex is a generalization, for weighted points, of the radius of the smallest ball enclosing the simplex. It is the dominant quantity in the definition of the order on simplices which induces the lexicographic order $\sqsubseteq_{lex}$ on chains. The proof then proceeds in 3 main steps:

1) For p large enough, the weight $w_p(\sigma)^p$ of any single simplex σ in the $\|\cdot\|_{(p)}$ norm is larger than the sum of all weights of all simplices with smaller bounding weight than σ (Lemma 4.5 in Section 4.1). This fact allows to focus on the link of a single simplex, the one with the largest bounding weight $\mu_B^{\neq}$ defined in (30), for which some simplices in the minimum of $\sqsubseteq_{lex}$ and $\sqsubseteq_p$ would differ as explained in Section 6.

2) We introduce (Sections 4.2 and 4.3) an extension of the classical lifted paraboloid construction that allows to see the simplicial structure of the link of a simplex τ in the Delaunay triangulation as a convex cone. In this representation, we study the subcomplex of this link corresponding to cofaces of τ with same bounding weight as τ. We show that this subcomplex is isomorphic to a set of bounded and visible faces of a convex polytope. It is therefore a simplicial ball (Lemma 4.13 in Section 4.3).

3) By induction on the dimension of convex cones and convex polytopes, one shows that this bounded subcomplex of the boundary of a convex polytope visible from the origin can be expressed as a the minimum under boundary constraint (Lemma 5.4 in Section 5) for another lexicographic order.

This second lexicographic order corresponds, on the restriction of the link of τ to simplices with the same bounding weight, to the first lexicographic order on corresponding full dimensional simplices in the star of τ (Lemmas 6.1 and 6.2 in Section 6).

3.2 Weighted points and weighted distances

We follow the terminology, notations and conventions from Section 4.4 of [3].

Definition 3.2 (Section 4.4 in [3])**.** *Given two weighted points $(P_1, \mu_1), (P_2, \mu_2) \in \mathbb{R}^n \times \mathbb{R}$ their weighted distance is defined as:*

$$\mathrm{D}\left((P_1, \mu_1), (P_2, \mu_2)\right) \underset{def.}{=} (P_1 - P_2)^2 - \mu_1 - \mu_2$$

We say (P_1, μ_1) and (P_2, μ_2) are orthogonal *if* $\mathrm{D}\left((P_1, \mu_1), (P_2, \mu_2)\right) = 0$.

We introduce a first generic condition:

Condition 3.1. *We say that $\mathbf{P} = \{(P_1, \mu_1), \ldots, (P_N, \mu_N)\} \subset \mathbb{R}^n \times \mathbb{R}$ satisfies the first generic condition if no $(n+1)$ points $\{P_{i_1}, \ldots, P_{i_{n+1}}\}$ lie on a same $(n-1)$-dimensional affine space.*

We now define the generalization of the circumsphere and the smallest enclosing ball for sets of weighted points.

Definition 3.3. *Assume that $\mathbf{P}$ satisfies Condition 3.1. Given a k-simplex $\sigma \subset \mathbf{P}$ with $0 \leq k \leq n$, the generalized circumsphere and smallest enclosing ball of σ are the weighted points $(P_C, \mu_C)(\sigma)$ and $(P_B, \mu_B)(\sigma)$ respectively defined as:*

$$\mu_C(\sigma) \underset{def.}{=} \min\left\{\mu \in \mathbb{R}, \exists P \in \mathbb{R}^n, \forall (P_i, \mu_i) \in \sigma,\ \mathrm{D}\left((P, \mu), (P_i, \mu_i)\right) = 0\right\} \tag{1}$$

$$\mu_B(\sigma) \underset{def.}{=} \min\left\{\mu \in \mathbb{R}, \exists P \in \mathbb{R}^n, \forall (P_i, \mu_i) \in \sigma,\ \mathrm{D}\left((P, \mu), (P_i, \mu_i)\right) \leq 0\right\} \tag{2}$$

$P_C(\sigma)$ and $P_B(\sigma)$ are respectively the unique points P that realize the minimum in Equations (1) and (2). The weights $\mu_C(\sigma)$ and $\mu_B(\sigma)$ are called respectively circumweight and bounding weight of σ. When $\forall i, \mu_i = 0$, they correspond respectively to the square of the circumradius and the square of the radius of the smallest ball enclosing σ.

The set $\{(P, \mu), \forall (P_i, \mu_i) \in \sigma, \mathrm{D}\left((P, \mu), (P_i, \mu_i)\right) = 0\}$, on which the first $\arg\min$ is taken, is not empty, thanks to the generic condition 3.1.

Lemma 3.4 (Proof in Appendix B)**.** *For any k-simplex $\sigma \in K_{\mathbf{P}}$, one has $P_B(\sigma) \in |\sigma|$.*

Condition 3.2. *We say that $\mathbf{P} = \{(P_1, \mu_1), \ldots, (P_N, \mu_N)\} \subset \mathbb{R}^n \times \mathbb{R}$ is in* general position *if it satisfies the first generic condition 3.1 and if, for a k-simplex σ and a k'-simplex σ' in $K_{\mathbf{P}}$ with $2 \leq k, k' \leq n$, one has:*

$$\mu_C(\sigma) = \mu_C(\sigma') \Rightarrow \sigma = \sigma' \tag{3}$$

From now on, we assume $\mathbf{P}$ to be in general position i.e. it satisfies Condition 3.2.

Lemma 3.5 (Proof in Appendix C)**.** *Under generic condition 3.2 on $\mathbf{P}$, for any simplex σ, there exists a unique inclusion minimal face $\Theta(\sigma)$ of σ such that $(P_B, \mu_B)(\sigma) = (P_C, \mu_C)(\Theta(\sigma))$. Moreover one has $(P_C, \mu_C)(\Theta(\sigma)) = (P_B, \mu_B)(\Theta(\sigma))$.*

Figure 1 illustrates the possibilities for $\Theta(\sigma)$ in the case $n = 3$ and zero weights.

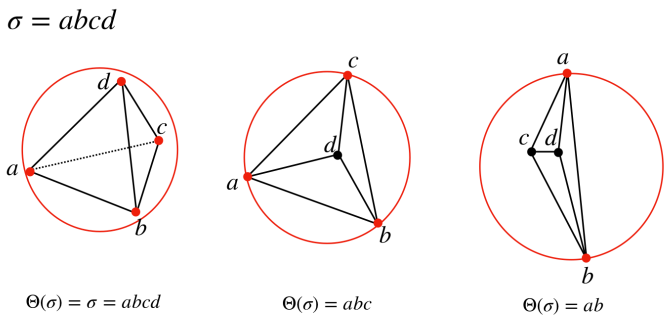

Figure 1: Illustration of the definition of $\Theta(\sigma)$ for a tetrahedron $\sigma = abcd$ in the case of zero weights.

3.3 Regular triangulation order on simplices

For a n-simplex σ, we define a $(k + \dim(\Theta(\sigma)))$-dimensional face $\Theta_k(\sigma)$ as follows. For $k = 0$, $\Theta_0(\sigma) = \Theta(\sigma)$, where $\Theta(\sigma)$ is defined in Lemma 3.5.

For $k > 0$, $\Theta_k(\sigma)$ is the $(\dim(\Theta_{k-1}(\sigma)) + 1)$-dimensional coface of $\Theta_{k-1}(\sigma)$ with minimal circumradius:

$$\Theta_k(\sigma) = \underset{\substack{\Theta_{k-1}(\sigma)\subset\tau\subset\sigma \\ \dim(\tau)=\dim(\Theta_{k-1}(\sigma))+1}}{\arg\min} \mu_C(\tau) \tag{4}$$

and $\mu_k(\sigma)$ is the circumweight of $\Theta_k(\sigma)$: $\mu_k(\sigma) = \mu_C(\Theta_k(\sigma))$. In particular, $\mu_0(\sigma) = \mu_C(\Theta(\sigma)) = \mu_B(\sigma)$ (by Lemma 3.5) and if $k = \dim(\sigma) - \dim(\Theta(\sigma))$ then $\mu_k(\sigma) = \mu_C(\sigma)$.

Observe that, thanks to generic condition 3.2 and Lemma 3.5, one has, for two n-simplices σ_1, σ_2: $\mu_B(\sigma_1) = \mu_B(\sigma_2) \Rightarrow \Theta(\sigma_1) = \Theta(\sigma_2)$ and therefore, if $\mu_B(\sigma_1) = \mu_B(\sigma_2)$, $\mu_k(\sigma_1)$ and $\mu_k(\sigma_2)$ are defined for the same range of values of k.

We define the following order relation on n-simplices (recall that $\mu_0(\sigma) = \mu_B(\sigma)$):

$$\sigma_1 \leq \sigma_2 \underset{\text{def.}}{\iff} \sigma_1 = \sigma_2 \quad \text{or} \quad \begin{cases} \mu_0(\sigma_1) < \mu_0(\sigma_2) \\ \quad \text{or} \\ \exists k \geq 1,\ \mu_k(\sigma_1) > \mu_k(\sigma_2) \\ \text{and } \forall j, 0 \leq j < k,\ \mu_j(\sigma_1) = \mu_j(\sigma_2) \end{cases} \tag{5}$$

One can check that when $\mathbf{P}$ is in general position, the relation $\leq$ is a total order.

For example, when $n = 2$ and the weights are zero, this order on triangles consists in first comparing the radii of the smallest circles enclosing the triangles $T_i, i = 1, 2$, whose squares are $\mathrm{R_B}(T_i)^2 = \mu_B(T_i) = \mu_0(T_i)$. This is generically enough for acute triangles, but not for obtuse triangles that could generically share their longest edge. In this case the tie is broken by comparing in reverse order the circumradii, whose squares are $\mathrm{R_C}(T_i)^2 = \mu_C(T_i) = \mu_1(T_i)$.

Following Definition 2.1 the order $\leq$ on n-simplices induces a lexicographic order $\sqsubseteq_{lex}$ on the n-chains of $K_{\mathbf{P}}$.

Observation 3.6. *From the definition of the order on simplices, the lexicographic minimum is invariant under a global translation of the weights by a common shift s: $\forall i, \mu_i \leftarrow \mu_i + s$. (as explicitly explained in observation 7.1 in Appendix A). The same holds for regular triangulations. Therefore proving Theorem 3.1 for non positive weights is enough to extend it to any weights.*

4 Regular triangulations of weighted points

We recall now the definition of a regular triangulation over a set of weighted points. Regular triangulation can alternatively be defined as the dual of the Laguerre (or power) diagram of a set of weighted points. We use here a generalization of the empty sphere property of Delaunay triangulations.

Definition 4.1 (Lemma 4.5 in [3])**.** *A regular triangulation $\mathcal{T}$ of the set of weighted points $\mathbf{P} = \{(P_1, \mu_1), \ldots, (P_N, \mu_N)\} \subset \mathbb{R}^n \times \mathbb{R}$, $N \geq n+1$, is a triangulation of the convex hull of $\{P_1, \ldots, P_N\}$ taking its vertices in $\{P_1, \ldots, P_N\}$ such that for any simplex $\sigma \in \mathcal{T}$, if $(P_C(\sigma), \mu_C(\sigma))$ is the generalized circumsphere of σ, then:*

$$(P_i, \mu_i) \in \mathbf{P} \setminus \sigma \Rightarrow \mathrm{D}\left((P_C(\sigma), \mu_C(\sigma)), (P_i, \mu_i)\right) > 0$$

4.1 Lift of weighted points and p-norms

Given a weighted point $(P, \mu) \in \mathbb{R}^n \times \mathbb{R}$, its lift with respect to an implicit origin $O \in \mathbb{R}^n$, denoted by $\mathrm{lift}(P, \mu)$, is a point in $\mathbb{R}^n \times \mathbb{R}$ given by:

$$\mathrm{lift}(P, \mu) \underset{\text{def.}}{=} ((P - O),\ (P - O)^2 - \mu)$$

Similarly to Delaunay triangulations, it is a well known fact that simplices of the regular triangulation of $\mathbf{P}$ are in one-to-one correspondence with the lower convex hull of $\mathrm{lift}(\mathbf{P})$:

Proposition 4.2. *A simplex σ is in the regular triangulation of $\mathbf{P}$ if and only if $\mathrm{lift}(\sigma)$ is a simplex on the lower convex hull of $\mathrm{lift}(\mathbf{P})$.*

Based on this lifted paraboloid formulation, the idea of variational formulation for Delaunay triangulations has emerged [6]. This idea has been exploited further in order to optimize triangulations in [5, 1]. We follow here the same idea but the variational formulation, while using the same criterion, is applied on the linear space of chains, which can be seen as a superset of the space of triangulations.

We define a function on the convex hull of a k-simplex $f_\sigma : |\sigma| \to \mathbb{R}$ where $\sigma = \{(P_0, \mu_0), \ldots, (P_k, \mu_k)\}$ as the difference between the linear interpolation of the height of the lifted vertices and the function $x \to (x - O)^2$. More precisely, for a point $x \in |\sigma|$ with barycentric coordinates $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$, we have $x = \sum_i \lambda_i P_i$ and:

$$f_\sigma : x \mapsto f_\sigma(x) \underset{\text{def.}}{=} \left(\sum_i \lambda_i ((P_i - O)^2 - \mu_i) \right) - (x - O)^2 \tag{6}$$

A short computation shows that the function f_σ, expressed in terms of barycentric coordinates, is invariant by isometry (translation, rotation or symmetry on σ). In particular $f_\sigma(x)$ does not depend on the origin O of the lift.

It follows from Proposition 4.2 that, if σ_{reg} is a simplex containing x in the regular triangulation of $\mathbf{P}$, for any other simplex σ containing x with vertices in $\mathbf{P}$:

$$f_{\sigma_{reg}}(x) \leq f_\sigma(x) \tag{7}$$

In the particular case where all weights are non positive, that is $\forall i,\ \mu_i \leq 0$, the convexity of $x \mapsto x^2$ says that the expression of $f_\sigma(x)$ in (6) is never negative and in this case (7) implies that defining the weight w_p of a n-simplex σ as:

$$w_p(\sigma) \underset{\text{def.}}{=} \|f_\sigma\|_p = \left(\int_{|\sigma|} f_\sigma(x)^p dx \right)^{\frac{1}{p}} \tag{8}$$

allows to characterize the regular triangulation as the one induced by the chain Γ_{reg} that, among all chains with boundary $\beta_{\mathbf{P}}$, minimizes:

$$\Gamma \to \|\Gamma\|_{(p)} \underset{\text{def.}}{=} \sum_\sigma |\Gamma(\sigma)| w_p(\sigma)^p \tag{9}$$

In this last equation, the notation $|\Gamma(\sigma)|$ instead of $\Gamma(\sigma)$ is there since $\Gamma(\sigma) \in \mathbb{Z}_2$ and the sum is in $\mathbb{R}$: $\Gamma(\sigma) \to |\Gamma(\sigma)|$ convert coefficients in $\mathbb{Z}_2$ into binary real numbers in $\{0, 1\}$.

Formally, we have the following Proposition 4.3 that characterizes regular triangulations as a linear programming problem over $\mathbb{Z}_2$.

Proposition 4.3 (Proof in Appendix D)**.** *Let $\mathbf{P} = \{(P_1, \mu_1), \ldots, (P_N, \mu_N)\} \subset \mathbb{R}^n \times \mathbb{R}$, with $N \geq n+1$ be in general position with non positive weights, and let $K_\mathbf{P}$ be the n-dimensional full complex over $\mathbf{P}$. Denote by $\beta_\mathbf{P} \in \boldsymbol{C_{n-1}}(K_\mathbf{P})$ the $(n-1)$-chain made of simplices belonging to the boundary of $\mathcal{CH}(\mathbf{P})$. For any $p \in [1, \infty)$, if:*

$$\Gamma_{reg} = \underset{\substack{\Gamma \in \boldsymbol{C_n}(K_\mathbf{P}) \\ \partial\Gamma = \beta_\mathbf{P}}}{\arg\min} \|\Gamma\|_{(p)}$$

then the simplicial complex $|\Gamma_{reg}|$ support of Γ_{reg} is the regular triangulation of $\mathbf{P}$.

Lemma 4.4 (Proof in Appendix E)**.** *One has:*

$$\sup_{x \in |\sigma|} f_\sigma(x) = \mu_B(\sigma) \tag{10}$$

The following is an immediate consequence used in the proof of Lemma 4.5:

$$\lim_{p \to \infty} w_p(\sigma) = w_\infty(\sigma) = \|f_\sigma\|_\infty = \sup_{x \in |\sigma|} f_\sigma(x) = \mu_B(\sigma) \tag{11}$$

Lemma 4.5 (Proof in Appendix F)**.** *Let $\mathbf{P} = \{(P_1, \mu_1), \ldots, (P_N, \mu_N)\} \subset \mathbb{R}^n \times \mathbb{R}$, with $N \geq n+1$, be in general position with non positive weights. Let $K_\mathbf{P}$ be the corresponding n-dimensional full complex. For p large enough, the weight $w_p(\sigma)^p$ of any n-simplex $\sigma \in K_\mathbf{P}$, is larger than the sum of all n-simplices in $K_\mathbf{P}$ with smaller bounding weight μ_B. In other words, if $K_\mathbf{P}^{[n]}$ is the set of n-simplices in $K_\mathbf{P}$:*

$$\exists p^\star, \forall p \geq p^\star, \forall \sigma \in K_\mathbf{P}^{[n]}, \quad w_p(\sigma)^p > \sum_{\tau \in K_\mathbf{P}^{[n]}, \mu_B(\tau) < \mu_B(\sigma)} w_p(\tau)^p$$

As explained in the proof of the main theorem of Section 6, Lemma 4.5 allows us to focus on the link of a single simplex τ. However, before that, we need to introduce geometrical constructions that give an explicit representation of this link (Sections 4.2 and 4.3).

4.2 Projection on the bisector of a simplex

We denote by $\mathbf{bis}_\tau$ the $(n-k)$-dimensional affine space *bisector* of τ, formally defined as:

$$\mathbf{bis}_\tau \underset{\text{def.}}{=} \{x \in \mathbb{R}^n, \forall v_1, v_2 \in \tau, \mathrm{D}((x, 0), v_1) = \mathrm{D}((x, 0), v_2)\} \tag{12}$$

In the particular case where $\dim(\tau) = 0$, one has $\mathbf{bis}_\tau = \mathbb{R}^n$.

Let $x \mapsto \pi_{\mathbf{bis}_\tau}(x)$ and $x \mapsto d(x, \mathbf{bis}_\tau) = d(x, \pi_{\mathbf{bis}_\tau}(x))$ denote respectively the orthogonal projection on and the minimal distance to $\mathbf{bis}_\tau$. We define a projection $\pi_\tau : \mathbf{P} \to \mathbf{bis}_\tau \times \mathbb{R}$ as follows:

$$\pi_\tau(P, \mu) \underset{\text{def.}}{=} \left(\pi_{\mathbf{bis}_\tau}(P), \mu - d(P, \mathbf{bis}_\tau)^2\right) \tag{13}$$

Figure 2 illustrates π_τ for ambient dimension 3 and $\dim(\tau) = 1$.

Let $o_\tau = P_C(\tau) \in \mathbf{bis}_\tau$ denote the (generalized) circumcenter of τ. If $(P_i, \mu_i) \in \tau$, then $\mathrm{D}((o_\tau, \mu_C(\tau)), (P_i, \mu_i)) = 0$.

Since $o_\tau = \pi_{\mathbf{bis}_\tau}(P_i)$ we have $(P_i - o_\tau)^2 - \mu_i - \mu_C(\tau) = d(P, \mathbf{bis}_\tau)^2 - \mu_i - \mu_C(\tau) = 0$. It follows that if we denote $\mu(\pi_\tau(P_i, \mu_i))$ the weight of $\pi_\tau(P_i, \mu_i)$, one has $\mu(\pi_\tau(P_i, \mu_i)) = \mu_i - d(P, \mathbf{bis}_\tau)^2 = -\mu_C(\tau)$. Therefore, π_τ sends all vertices of τ to a single weighted point:

$$(P_i, \mu_i) \in \tau \Rightarrow \pi_\tau(P_i, \mu_i) = (o_\tau, -\mu_C(\tau)) \tag{14}$$

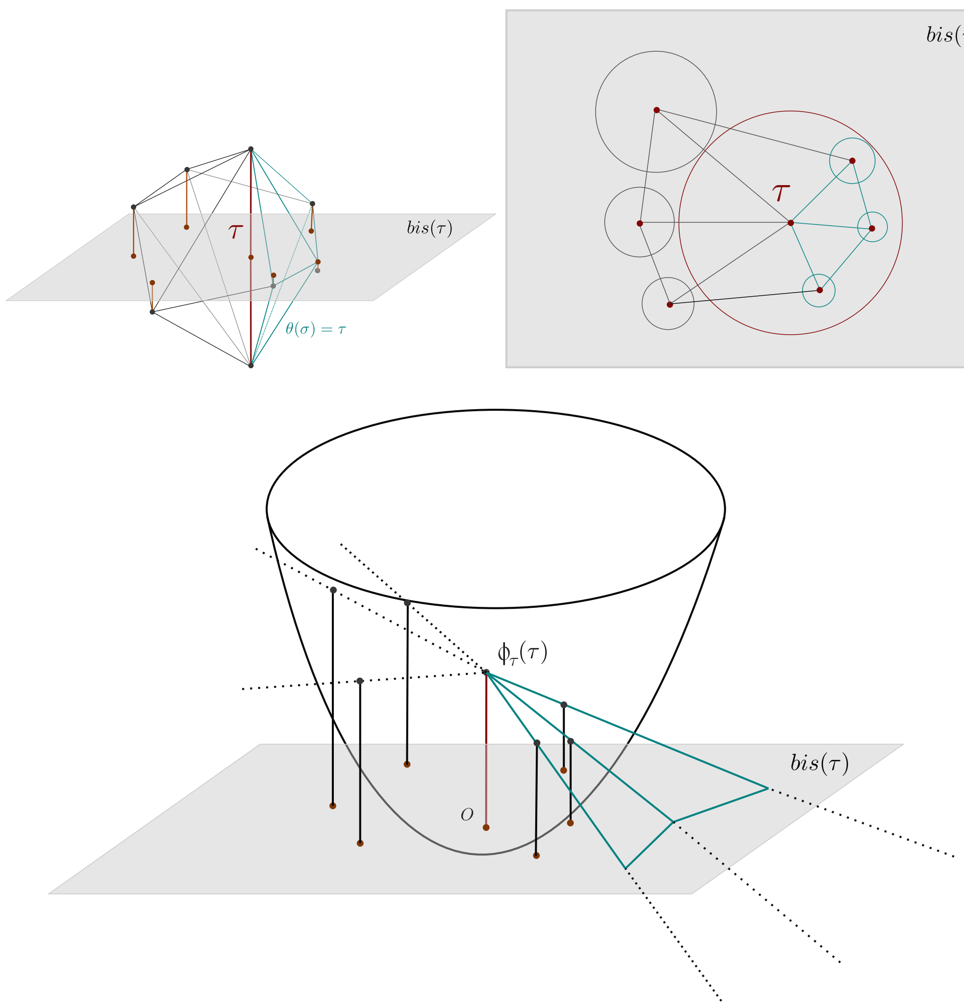

Figure 2: Illustration for the definition of $\mathbf{bis}_\tau$, π_τ (top left and right) and Φ_τ (bottom).

Lemma 4.6 (Proof in Appendix G)**.** *Let* $\mathbf{P}$ *be in general position,* $\tau \in K_{\mathbf{P}}$ *a* k*-simplex and* $\sigma \in K_{\mathbf{P}}$ *a coface of* τ*. Then* σ *is in the regular triangulation of* $\mathbf{P}$ *if and only if* $\pi_\tau(\sigma)$ *is a coface of the vertex* $\{(o_\tau, -\mu_C(\tau))\} = \pi_\tau(\tau)$ *in the regular triangulation of* $\pi_\tau(\mathbf{P})$*.*

An immediate consequence of Lemma 4.6 is:

Corollary 4.7. *The projection* π_τ *preserves the structure of the regular triangulation around* τ*, more precisely:*

1. *the simplex* τ *is in the regular triangulation of* $\mathbf{P}$ *if and only if the vertex* $\pi_\tau(\tau) = \{(o_\tau, -\mu_C(\tau))\}$ *is a vertex of the regular triangulation of* $\pi_\tau(\mathbf{P})$*,*
2. *if* τ *is in the regular triangulation of* $\mathbf{P}$*,* π_τ *induces a simplicial isomorphism between the link of* τ *in the regular triangulation of* $\mathbf{P}$ *and the link of vertex* $\pi_\tau(\tau)$ *in the regular triangulation of* $\pi_\tau(\mathbf{P})$*.*

4.3 Polytope and shadow associated to a link in a regular triangulation

In this section, we study the link of a k-simplex τ in the regular triangulation of $\mathbf{P}$ that satisfies:

$$(P_B, \mu_B)(\tau) = (P_C, \mu_C)(\tau) \tag{15}$$

We know from Proposition 4.2 that the link of τ in the regular triangulation is isomorphic to the link of $\mathrm{lift}(\pi_\tau(\tau))$ on the boundary of the lower convex hull of $\mathrm{lift}(\pi_\tau(\mathbf{P}))$. We consider the lift with the origin at o_τ, in other words, the image of a vertex $(P, \mu) \in \mathbf{P}$ is:

$$\begin{aligned} \Phi_\tau(P,\mu) &\underset{\text{def.}}{=} \mathrm{lift}\left(\pi_\tau(P,\mu)\right) \\ &= \left(\pi_{\mathbf{bis}_\tau}(P) - o_\tau,\ (\pi_{\mathbf{bis}_\tau}(P) - o_\tau)^2 - \mu + d(P, \mathbf{bis}_\tau)^2\right) \end{aligned}$$

Observe that:

$$\Phi_\tau(\tau) = \{(0, \mu_C(\tau)\}$$

Call P_τ the set of weighted points in $(P_i, \mu_i) \in \mathbf{P}$ such that $\tau \cup (P_i, \mu_i)$ has same bounding weight as τ. In the case of the Delaunay triangulation, P_τ corresponds to the set of points in $\mathbf{P}$ inside the smallest ball enclosing τ. Formally, using the assumption (15):

$$P_\tau \underset{\text{def.}}{=} \{(P_i, \mu_i) \in \mathbf{P} \setminus \tau,\ \mathrm{D}\left((P_C(\tau), \mu_C(\tau)), (P_i, \mu_i)\right) < 0\} \tag{16}$$

Denote by K_τ the $(n-k-1)$-dimensional simplicial complex made of all up to dimension $(n-k-1)$ simplices over vertices in P_τ. Observe that:

$$\begin{aligned} &\mathrm{D}\left((P_C(\tau), \mu_C(\tau)), (P_i, \mu_i)\right) < 0 \\ \iff &\left(\pi_{\mathbf{bis}_\tau}(P_i) - o_\tau\right)^2 - \mu_i + d\left(P_i, \mathbf{bis}_\tau\right)^2 < \mu_C(\tau) \end{aligned} \tag{17}$$

Denote by $\mathrm{Height}(\mathrm{lift}((P, \mu))$ the *height* of the lift of a point (P, μ), defined as the last coordinate of the lift, so that:

$$\mathrm{Height}(\Phi_\tau(P, \mu)) = (\pi_{\mathbf{bis}_\tau}(P) - o_\tau)^2 - \mu + d(P, \mathbf{bis}_\tau)^2 \tag{18}$$

Since under our generic conditions we have $\pi_{\mathbf{bis}_\tau}(P) - o_\tau \neq 0$, (16), (17) and (18) imply that $\exists v \in P_\tau \Rightarrow \mathrm{Height}(\Phi_\tau(v, \mu)) > 0$ and (17) can be rephrased as:

Observation 4.8. *A vertex belongs to* P_τ *if and only if the height of its image by* Φ_τ *is strictly less than* $\mu_C(\tau) > 0$*:*

$$v \in P_\tau \iff 0 < \mathrm{Height}(\Phi_\tau(v)) < \mu_C(\tau)$$

This observation allows to define the following conical projection:

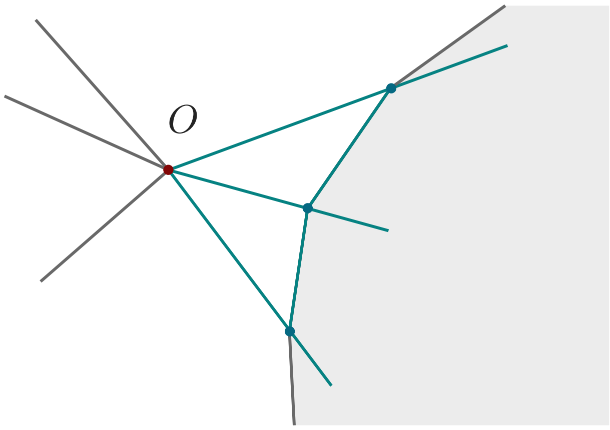

Figure 3: Illustration of the shadow polytope (grey area) of Definition 4.11 corresponding to the example of Figure 2.

Definition 4.9 (Shadow)**.** *Let v be a vertex in P_τ. The* shadow $\mathrm{Sh}_\tau(v)$ *of v is a point in the $(n-k)$-dimensional Euclidean space* $\mathbf{bis}_\tau$ *defined as the intersection of the half-line starting at $(0, \mu_C(\tau))$ and going through $\Phi_\tau(v)$ with the space* $\mathbf{bis}_\tau$*.*

$$\mathrm{Sh}_\tau(P,\mu) \underset{def.}{=} \frac{\mu_C(\tau)}{\mu_C(\tau) - \mathrm{Height}(\Phi_\tau(P,\mu))}(\pi_{\mathbf{bis}_\tau}(P) - o_\tau)$$

The shadow of a simplex $\sigma \in K_\tau$ is a simplex in $\mathbf{bis}_\tau$ *whose vertices are the shadows of vertices of σ.*

Let Γ_{reg} be the n-chain containing the n-simplices of the regular triangulation of $\mathbf{P}$ and $|\Gamma_{reg}|$ the corresponding simplicial complex. Denote by $\mathrm{X}(\tau) \in \boldsymbol{C_{n-k-1}}(K_\tau)$ the $(n-k-1)$-chain made of simplices $\sigma \in K_\tau$ such that $\tau \cup \sigma \in |\Gamma_{reg}|$:

$$\mathrm{X}(\tau) \underset{\text{def.}}{=} \{\sigma \in K_\tau,\ \dim(\sigma) = n-k-1,\ \tau \cup \sigma \in \Gamma_{reg}\} \tag{19}$$

In the following, we call *polytope* a finite intersection of closed half spaces $\cap_i H_i$. The *convex cone* of a polytope at a point p is the intersection of all such H_i whose boundary contain p.

Definition 4.10 (Polytope facet **visible** from the point 0)**.** *We say that a facet f of a polytope is* visible from the point 0*, or* visible *for short, if the closed half-space H containing the polytope and whose boundary is the supporting plane of f does not contains 0.*

Definition 4.11 (Shadow Polytope)**.** *The (possibly empty) intersection of the convex cone of the lower convex hull of $\Phi_\tau(\mathbf{P})$ at $\Phi_\tau(\tau) = (0, \mu_C(\tau))$ with* $\mathbf{bis}_\tau$ *is called* shadow polytope *of τ.*

Figure 3 depicts the shadow polytope corresponding to the example of Figure 2 as the hatched area and the bounded cells of its boundary by the two edges in blue.

For each upper half-space H_j contributing to the convex cone of the lower convex hull of $\Phi_\tau(\mathbf{P})$ at $(0, \mu_C(\tau))$, the intersection $H_j \cap \mathbf{bis}_\tau$ is a $(n-k)$-dimensional half-space in $\mathbf{bis}_\tau$. The shadow polytope is precisely defined as the intersection of all such half-spaces $H_j \cap \mathbf{bis}_\tau$. Since each H_j is a upper half-space and since by observation 4.8 one has $\mu_C(\tau) > 0$, H_j does not contains $(0,0)$, which implies that $H_j \cap \mathbf{bis}_\tau$ does not contain the point o_τ in $\mathbf{bis}_\tau$. It follows that:

Observation 4.12. *All facets of the shadow polytope are visible from* 0*.*

Lemma 4.13 (Proof in Appendix H)**.** *Let $\mathbf{P}$ be in general position with non positive weights.*

1. *τ is in the regular triangulation of $\mathbf{P}$ if and only if $\Phi_\tau(\tau) = (0, \mu_C(\tau))$ is an extremal point of the convex hull of $\Phi_\tau(\mathbf{P})$.*

2. *When τ is in the regular triangulation of $\mathbf{P}$, its link is isomorphic to the link of the vertex $\Phi_\tau(\tau) = (0, \mu_C(\tau))$ in the simplicial complex corresponding to the boundary of the lower convex hull of $\Phi_\tau(\mathbf{P})$.*

3. *When τ is in the regular triangulation of $\mathbf{P}$, Sh_τ induces a bijection between the simplices in $\mathrm{X}(\tau)$ and the set of bounded facets of the boundary of the shadow polytope.*

5 Convex cone as lexicographic minimal chain

This section is self-contained and does not relies on previous constructions. Lemma 5.4 is a key ingredient of the proof of Theorem 3.1 but is also a result of independent interest: visible convex hulls can be defined as minimal lexicographic chains. Lemmas 5.2 and 5.3 (Proof in Appendix I and J) will be instrumental in the proof of Lemma 5.4.

Definition 5.1 (Trace of a chain in a link)**.** *Given a k-simplex τ in a simplical complex K and a n-chain on K for $n > k$, we call* trace of Γ on the link of τ *the $(n-k-1)$-chain $\mathrm{Tr}_\tau(\Gamma)$ defined in the link of τ by:*

$$\mathrm{Tr}_\tau(\Gamma)(\sigma) \underset{def.}{=} \Gamma(\tau \cup \sigma)$$

Lemma 5.2 (Proof in Appendix I)**.** *Given a k-simplex τ in a simplical complex K and a n-chain Γ on K for $n > k$, one has:*

$$\partial\,\mathrm{Tr}_\tau(\Gamma) = \mathrm{Tr}_\tau(\partial\Gamma)$$

Lemma 5.3 (Proof in Appendix J)**.** *Let $C \subset \mathbb{R}^n$ be a polytope and $O \in \mathbb{R}^n \setminus C$. Let $X \subset \partial C$ be a compact set union of facets of C visible from O. If $x \in X$ maximizes the distance to O, then x is in the closure of $\partial C \setminus X$.*

We need to define another order on simplices together with its induced order on chains, respectively denoted $\leq_{Sh}$ and $\sqsubseteq_{\mathrm{Sh}}$. We associate to a $(n-1)$-simplex σ in $\mathbb{R}^n$ that does not contain 0 a dimension increasing sequence of faces $\varnothing = \tau_{-1}(\sigma) \subset \tau_0(\sigma) \subset \ldots \subset \tau_{n-1}(\sigma) = \sigma$ with $\dim(\tau_i) = i$. Under a simple generic condition, it is defined as follows.

$\tau_{-1}(\sigma) = \varnothing$ and $\tau_0(\sigma)$ is the vertex of σ farthest from 0. More generally, define the distance from a flat (an affine space) F to 0 as $d_0(F) = \inf_{p\in F} d(p, 0)$. If ζ is a non degenerate i-simplex for $i \geq 0$, defines $d_0(\zeta) = d_0(F(\zeta))$ where $F(\zeta)$ is the i-dimensional flat support of ζ. For $i \geq 0$, $\tau_i(\sigma)$ is the coface of dimension i of $\tau_{i-1}(\sigma)$ whose supporting i-flat is farthest from 0:

$$\tau_i(\sigma) \underset{\text{def.}}{=} \arg \max_{\substack{\zeta \supset \tau_{i-1}(\sigma) \\ \dim(\zeta)=i}} d_0(\zeta) \tag{20}$$

For $i = 0, \ldots, n-1$ we set $\delta_i(\sigma) = d_0(\tau_i(\sigma))$ and the comparison $<_{Sh}$ between two $(n-1)$-simplices σ_1 and σ_2 is a lexicographic order on the sequences $(\delta_i(\sigma_1))_{i=0,\ldots n-1}$ and $(\delta_i(\sigma_2))_{i=0,\ldots n-1}$:

$$\sigma_1 <_{Sh} \sigma_2 \underset{\text{def.}}{\iff} \begin{cases} \exists k \geq 0,\ \delta_k(\sigma_1) < \delta_k(\sigma_2) \\ \text{and } \forall j, 0 \leq j < k,\ \delta_j(\sigma_1) = \delta_j(\sigma_2) \end{cases} \tag{21}$$

which defines an order relation:

$$\sigma_1 \leq_{Sh} \sigma_2 \underset{\text{def.}}{\iff} \sigma_1 = \sigma_2 \quad \text{or} \quad \sigma_1 <_{Sh} \sigma_2 \tag{22}$$

Condition 5.1. *Let K be a $(n-1)$-dimensional simplicial complex. For any pair of simplices $\sigma_1, \sigma_2 \in K$: $\dim(\sigma_1) = \dim(\sigma_2) = k$ and $d_0(\sigma_1) = d_0(\sigma_2) \Rightarrow \sigma_1 = \sigma_2$.*

Under Condition 5.1, $\leq_{Sh}$ is a total order on simplices and, following Definition 2.1, the order $\leq_{Sh}$ on simplices induces a lexicographic order $\sqsubseteq_{\mathrm{Sh}}$ on k-chains of K.

Lemma 5.4. *Let P be a set of points in $\mathbb{R}^n$ such that $0 \in \mathbb{R}^n$ is not in the convex hull of P. Let K be the complete $(n-1)$-dimensional simplicial complex over P, i.e. the simplicial complex made of all $(n-1)$-simplices whose vertices are points in P with all their faces. Assume that K satisfies the generic condition 5.1.*

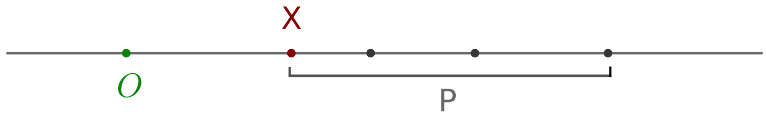

Figure 4: Illustration of Lemma 5.4 for $n = 1$.

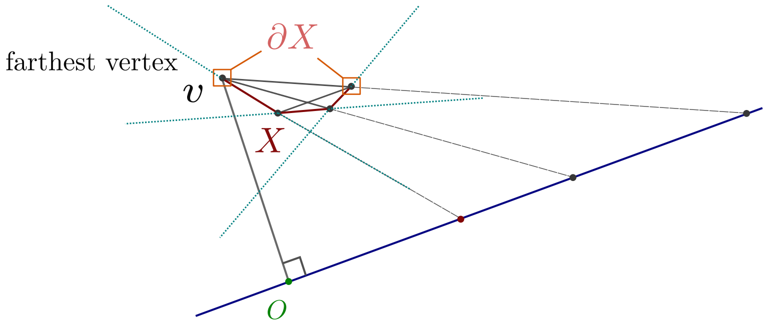

Figure 5: Illustration of the recursion in the proof of Lemma 5.4 for $n = 2$.

Let X be a $(n-1)$-chain in K whose $(n-1)$-simplices are on the boundary of the convex hull of P and are all visible *from $0 \in \mathbb{R}^n$. Then:*

$$X = \min_{\sqsubseteq_{\mathrm{Sh}}}\{\Gamma \in \boldsymbol{C_{n-1}}(K), \partial\Gamma = \partial X\} \tag{23}$$

where when $n = 1$ the boundary operator in (23) is meant as the boundary operator of reduced homology, i.e the linear operator $\tilde{\partial}_0 : \boldsymbol{C_{n-1}}(K) \to \mathbb{Z}_2$ that counts the parity.

The lemma is illustrated for $n = 2, 3$ on figures 5 and 6.

Proof. We first claim that the lemma holds for $n = 1$. In this case the fact that 0 is not in the convex hull of P means that the 1-dimensional points in P are either all positive, either all negative. The single simplex in the convex hull boundary visible from 0 is the point in P closest to 0, i.e. the one with the smallest absolute value, which corresponds to the minimum chain with odd parity in the $\sqsubseteq_{\mathrm{Sh}}$ order, which proves the claim.

We assume then the theorem to be true for the dimension $n - 1$ and proceed by induction. This recursion is illustrated on figure 5 for $n = 2$ and figure 6 for $n = 3$.

Consider the minimum:

$$\Gamma_{\min} = \min_{\sqsubseteq_{\mathrm{Sh}}}\{\Gamma \in \boldsymbol{C_{n-1}}(K), \partial\Gamma = \partial X\} \tag{24}$$

We need to prove that $\Gamma_{\min} = X$.

Let v be the (unique) vertex in the simplices of ∂X which is farthest from 0. Since v is a vertex in at least one simplex in $\partial X = \partial\Gamma_{\min}$, it must be a vertex in some simplex in $\Gamma_{\min}$.

Thanks to Lemma 5.3, if a point x is a local maximum in X of the distance to 0 one has $x \in \partial X$. It follows that v is also the vertex in the simplices of X which is farthest from 0.

Since v is the vertex in X farthest from 0 and since by definition $\Gamma_{\min} \sqsubseteq_{\mathrm{Sh}} X$, we know that $\Gamma_{\min}$ does not contain any vertex farther from the origin than v, therefore v is also the vertex in the simplices of $\Gamma_{\min}$ farthest from 0.

Since $\partial\Gamma_{\min} = \partial X$, Lemma 5.2 implies that:

$$\partial \operatorname{Tr}_v(\Gamma_{\min}) = \operatorname{Tr}_v(\partial\Gamma_{\min}) = \operatorname{Tr}_v(\partial X) \tag{25}$$

In order to define a lexicographic order on chains on the link of v in K, we consider the hyperplane Π containing 0 and orthogonal to the line $0v$. We associate to any $(n-2)$-simplex $\eta \in \mathrm{Lk}_K(v)$ the $(n-2)$ simplex $\pi_{v\Pi}(\eta)$ conical projection of η on Π with center v. In other words, if u is a vertex of η:

$$\{\pi_{v\Pi}(u)\} = \Pi \cap uv$$

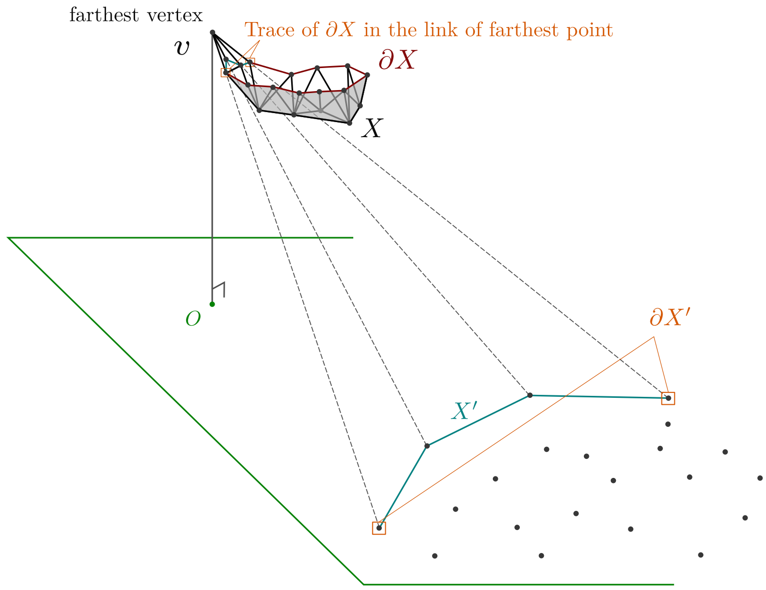

Figure 6: Illustration of the recursion in the proof of Lemma 5.4 for $n = 3$.

where uv denote the line going through u and v. The map $\pi_{v\Pi}$ is a conical projection on vertices but it extends to a bijection on simplices and an isomorphism on chains that trivially commutes with the boundary operator.

By definition of the lexicographic order $\sqsubseteq_{\mathrm{Sh}}$, the comparison of two chains whose farthest vertex is v starts by comparing their restrictions to the star of v. Therefore, since v is the farthest vertex in $\Gamma_{\min}$, the restriction of $\Gamma_{\min}$ to the star of v must be minimum under the constraint $\partial\Gamma = \partial X$. The constraint $\partial\Gamma = \partial X$ for the restriction of $\Gamma_{\min}$ to the star of v is equivalent to the constraint given by equation (25) or equivalently by:

$$\partial\pi_{v\Pi}\left(\mathrm{Tr}_v(\Gamma_{\min})\right) = \pi_{v\Pi}\left(\mathrm{Tr}_v(\partial X)\right)$$

and the minimization on the restriction of the $(n-1)$-chain $\Gamma_{\min}$ to the star of v can equivalently be expressed as the minimization of the $(n-2)$-chain $\gamma_{\min} = \pi_{v\Pi}\left(\mathrm{Tr}_v(\Gamma_{\min})\right)$ under the constraint $\partial\gamma_{\min} = \pi_{v\Pi}\left(\mathrm{Tr}_v(\partial X)\right)$, we have then:

$$\begin{aligned}
\gamma_{\min} &= \pi_{v\Pi}\left(\mathrm{Tr}_v\left(\Gamma_{\min}\right)\right) \\
&= \pi_{v\Pi}\left(\mathrm{Tr}_v\left(\min_{\sqsubseteq_{\mathrm{Sh}}}\{\Gamma \in \boldsymbol{C_{n-1}}(K), \partial\Gamma = \partial X\}\right)\right) \\
&= \min_{\sqsubseteq_{\mathrm{Sh}}}\{\gamma \in \boldsymbol{C_{n-2}}\left(\pi_{v\Pi}\left(\mathrm{Lk}_K(v)\right)\right), \partial\gamma = \pi_{v\Pi}\left(\mathrm{Tr}_v(\partial X)\right)\} \\
&= \min_{\sqsubseteq_{\mathrm{Sh}}}\{\gamma \in \boldsymbol{C_{n-2}}\left(\pi_{v\Pi}\left(\mathrm{Lk}_K(v)\right)\right), \partial\gamma = \partial\pi_{v\Pi}\left(\mathrm{Tr}_v(X)\right)\}
\end{aligned} \tag{26}$$

In the third equality of (26) we have used the fact that the orders on $(n-1)$-simplices in the star of v in K and the order on corresponding $(n-2)$-simplices in the image by $\pi_{v\Pi}$ of the link of v are compatible.

Indeed, if F is a k-flat in $\mathbb{R}^n$ going through v, we have (see Figure 7):

$$d_0(\pi_{v\Pi}(F)) = d_0(F \cap \Pi) = \frac{d_0(F)\|v - 0\|}{\sqrt{(v-0)^2 - d_0(F)^2}} \tag{27}$$

with the convention $d_0(\pi_{v\Pi}(F)) = +\infty$ in the non generic case where $F \cap \Pi = \varnothing$ (the denominator vanishes in this case while since $v \in F$ one has $F \cap \Pi \neq \varnothing \Rightarrow d_0(F) < \|v - 0\|$).

As seen on (27) $d_0(F) \mapsto d_0(\pi_{v\Pi}(F))$ is an increasing function and the orders are therefore consistent along the induction.

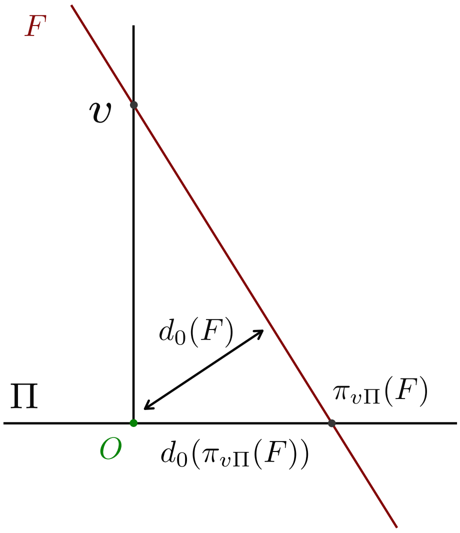

Figure 7: Illustration for Equation (27).

We claim that the minimization problem in the last member of (26) satisfies the condition of the theorem for $n' = n - 1$ which is assumed true by induction.

The recursion is as follows: hyperplane Π corresponds to $\mathbb{R}^{n'}$ with $n' = n - 1$ and:

- $n' \leftarrow n - 1$
- $P' \leftarrow \pi_{v\Pi}(P \setminus \{v\})$
- $K' \leftarrow \pi_{v\Pi}(\mathrm{Lk}_K(v))$
- $X' \leftarrow \pi_{v\Pi}\left(\mathrm{Tr}_v(X)\right)$

Since $(n-1)$-simplices in X are in convex positions, hyperplanes supporting these simplices, in particular the simplices in the star of v, separate all points of P from 0. It follows that the intersection of theses hyperplanes with the horizontal hyperplane, i.e. the images by $\pi_{v\Pi}$ of the hyperplanes, separates $P' = \pi_{v\Pi}(P \setminus \{v\})$ from 0. It follows that the $(n'-1)$-simplices in $X' = \pi_{v\Pi}\left(\mathrm{Tr}_v(X)\right)$ are in convex position and are visible from 0.

Therefore one can apply our lemma recursively, which gives us, using (26):

$$\gamma_{\min} = \pi_{v\Pi}\left(\mathrm{Tr}_v\left(\Gamma_{\min}\right)\right) = \pi_{v\Pi}\left(\mathrm{Tr}_v(X)\right)$$

It follows that the faces in the star of v corresponding to $\mathrm{Tr}_v\left(\Gamma_{\min}\right)$ belong to X. Call Y the $(n-1)$-chain made of these simplices in the star of v. We have both $Y \subset X$ and $Y \subset \Gamma_{\min}$. Since v is the vertex farthest from 0 in both X and in $\Gamma_{\min}$ one has by definition of the lexicographic order:

$$\begin{aligned}\Gamma_{\min} &= \min_{\sqsubseteq_{lex}}\{\Gamma \in \boldsymbol{C_{n-1}}(K), \partial\Gamma = \partial X\} \\ &= Y + \min_{\sqsubseteq_{lex}}\{\Gamma \in \boldsymbol{C_{n-1}}(K), \partial\Gamma = \partial(X - Y)\}\end{aligned}$$

So, by considering the new problem $X \leftarrow (X - Y)$ and iterating as long as X is not empty, we get our final result $\Gamma_{\min} = X$. $\square$

6 Proof of Theorem 3.1

The next two lemmas establish the connexion between the orders $\sqsubseteq_{lex}$ and $\sqsubseteq_{\mathrm{Sh}}$.

Lemma 6.1 (Proof in Appendix K). *For $\sigma_1, \sigma_2 \in K_\tau$, one has:*

$$\mu_C(\tau \cup \sigma_1) \geq \mu_C(\tau \cup \sigma_2) \iff d_0(\mathrm{Sh}_\tau(\sigma_1)) \leq d_0(\mathrm{Sh}_\tau(\sigma_2)) \tag{28}$$

For a n-chain Γ denote by $\downarrow_\rho \Gamma$ the chain obtained by removing from Γ all simplices with bounding weight strictly greater than ρ.

$$\downarrow_\rho \Gamma \underset{\text{def.}}{=} \{\sigma \in \Gamma, \mu_B(\sigma) \leq \rho\} \tag{29}$$

Lemma 6.2 (Proof in Appendix L). *For two n-chains $\Gamma_1, \Gamma_2 \in \boldsymbol{C_n}(K_{\mathbf{P}})$ if $\Gamma_1 \neq \Gamma_2$ one has:*

$$\begin{aligned} \downarrow_\rho \Gamma_1 \quad &\sqsubseteq_{lex} \quad \downarrow_\rho \Gamma_2 \\ \Rightarrow \mathrm{Sh}_{\tau_\rho}\left(\mathrm{Tr}_{\tau_\rho}\left(\downarrow_\rho \Gamma_1\right)\right) \quad &\sqsubseteq_{\mathrm{Sh}} \quad \mathrm{Sh}_{\tau_\rho}\left(\mathrm{Tr}_{\tau_\rho}\left(\downarrow_\rho \Gamma_2\right)\right) \end{aligned}$$

Proof of Theorem 3.1. We prove Theorem 3.1 in the case of non positive weights which then extends to any weights thanks to Observations 3.6 and 7.1. As in Proposition 4.3, denote by Γ_{reg} the chain that defines the regular triangulation of $\mathcal{CH}(\mathbf{P})$. As in Theorem 3.1 denote by $\beta_{\mathbf{P}} \in \boldsymbol{C_{n-1}}(K_{\mathbf{P}})$ the $(n-1)$-chain made of simplices belonging to the boundary of $\mathcal{CH}(\mathbf{P})$.

According to Proposition 4.3, Γ_{reg} minimizes $\Gamma \mapsto \|\Gamma\|_{(p)}$ among the chains with boundary $\beta_{\mathbf{P}}$ for any $p \geq 1$. In particular, Γ_{reg} minimizes $\Gamma \mapsto \|\Gamma\|_{p^\star}$ for the value $p^\star$ of Lemma 4.5.

Proposition 4.3 and Theorem 3.1 consider a minimum with respect to the same boundary condition while their objective differ. In order to prove Theorem 3.1, we have to show that both minimum still agree. By contradiction, we assume now that they differ, which means that, $\Gamma_{\min} \neq \Gamma_{reg}$ where $\Gamma_{\min}$ is the minimal chain of Theorem 3.1. Consider $\mu_B^{\neq}$ to be the largest bounding weight for which some simplex in $\Gamma_{\min}$ and Γ_{reg} differ:

$$\mu_B^{\neq} \underset{\text{def.}}{=} \max\{\mu_B(\sigma), \sigma \in \Gamma_{\min} + \Gamma_{reg}\} \tag{30}$$

There must be at least one simplex with bounding weight $\mu_B^{\neq}$ in Γ_{reg} as otherwise, by definition of $\mu_B^{\neq}$, there would be a simplex with radius $\mu_B^{\neq}$ in $\Gamma_{\min}$ and this would give $\Gamma_{reg} \sqsubseteq_{lex} \Gamma_{\min}$ with $\Gamma_{reg} \neq \Gamma_{\min}$ and since $\partial\Gamma_{reg} = \partial\Gamma_{\min} = \beta_{\mathbf{P}}$ this contradicts the definition of $\Gamma_{\min}$.

Similarly, it follows from Lemma 4.5 that if there was no simplex with bounding weight $\mu_B^{\neq}$ in $\Gamma_{\min}$, one would have $\|\Gamma_{\min}\|_{p^\star} < \|\Gamma_{reg}\|_{p^\star}$ and $\partial\Gamma_{reg} = \partial\Gamma_{\min}$: a contradiction with the minimality of Γ_{reg} for norm $\|\cdot\|_{p^\star}$ (Proposition 4.3). We have shown that if they differ, both Γ_{reg} and $\Gamma_{\min}$ must contain at least one simplex with bounding weight $\mu_B^{\neq}$.

We know from Lemma 3.5 and the generic condition 3.2 that the set of simplices with bounding weight $\mu_B^{\neq}$ are all cofaces of some unique dimension minimal simplex $\tau_{\mu_B^{\neq}}$. If $\dim(\tau_{\mu_B^{\neq}}) = n$, $\tau_{\mu_B^{\neq}}$ is the unique simplex in $K_{\mathbf{P}}$ whose bounding weight is $\mu_B^{\neq}$. But then Γ_{reg} and $\Gamma_{\min}$ coincide on simplices with bounding weight $\mu_B^{\neq}$, a contradiction with the definition of $\mu_B^{\neq}$.

Assume now that $\dim(\tau_{\mu_B^{\neq}}) = k < n$. By definition of $\mu_B^{\neq}$ one has:

$$\downarrow_{\mu_B^{\neq}} \Gamma_{reg} - \downarrow_{\mu_B^{\neq}} \Gamma_{\min} = \Gamma_{reg} - \Gamma_{\min} \tag{31}$$

where $\downarrow_{\mu_B^{\neq}}$ is defined in (29). In order to spare our eyes, we allow ourselves to replace for the rest of the section $\downarrow_{\mu_B^{\neq}}$ by $\downarrow$ and $\tau_{\mu_B^{\neq}}$ by τ. It follows from (31) that:

$$\partial\left(\downarrow \Gamma_{reg} - \downarrow \Gamma_{\min}\right) = \partial\Gamma_{reg} - \partial\Gamma_{\min} = \beta_{\mathbf{P}} - \beta_{\mathbf{P}} = 0$$

$\downarrow \Gamma_{reg}$ and $\downarrow \Gamma_{\min}$ have therefore the same boundary and by Lemma 5.2, their trace also have the same boundary:

$$\partial\, \mathrm{Tr}_\tau\left(\downarrow \Gamma_{min}\right) = \partial\, \mathrm{Tr}_\tau\left(\downarrow \Gamma_{reg}\right) \tag{32}$$

Observe that $\mathrm{Tr}_\tau(\downarrow \Gamma_{reg})$ coincide with the definition of $\mathrm{X}(\tau)$ in (19). We know then from Lemma 4.13 that the shadows of simplices in $\mathrm{X}(\tau) = \mathrm{Tr}_\tau(\downarrow \Gamma_{reg})$, that is $\mathrm{Sh}_\tau(\mathrm{X}(\tau)) = \mathrm{Sh}_\tau(\mathrm{Tr}_\tau(\downarrow \Gamma_{reg}))$ is a chain in $\mathrm{Sh}_\tau(K_\tau)$ made of the faces of the convex hull of $\mathrm{Sh}_\tau(P_\tau)$ visible from the origin 0.

In the remaining of this proof we use the lexicographic order $\sqsubseteq_{\mathrm{Sh}}$ on shadows of $(n-k-1)$-chains in K_τ, defined at the beginning of Section 5. This order is equivalent to the order $\sqsubseteq_{lex}$ on corresponding n-chains restricted to the set of simplices with bounding weight $\mu_B^{\neq}$ (Lemmas 6.1 and 6.2). This correspondence allows to conclude the proof by applying Lemma 5.4 that says the chain defined by visible faces of a polytope minimises the lexicographic order $\sqsubseteq_{\mathrm{Sh}}$ among chains with same boundary. More formally, thanks to Lemma 6.2 we have:

$$\downarrow \Gamma_1 \sqsubseteq_{lex} \downarrow \Gamma_2 \;\Rightarrow\; \mathrm{Sh}_\tau\left(\mathrm{Tr}_\tau\left(\downarrow \Gamma_1\right)\right) \sqsubseteq_{\mathrm{Sh}} \mathrm{Sh}_\tau\left(\mathrm{Tr}_\tau\left(\downarrow \Gamma_2\right)\right)$$

It follows that $\mathrm{Sh}_\tau\left(\mathrm{Tr}_\tau\left(\downarrow \Gamma_{min}\right)\right)$ is, among all chains in the complex $\mathrm{Sh}_\tau\left(K_\tau\right)$, the one that minimises $\sqsubseteq_{\mathrm{Sh}}$ under the constraint (32), or equivalently:

$$\partial\, \mathrm{Sh}_\tau\left(\mathrm{Tr}_\tau\left(\downarrow \Gamma_{min}\right)\right) = \partial\, \mathrm{Sh}_\tau\left(\mathrm{Tr}_\tau\left(\downarrow \Gamma_{reg}\right)\right)$$

Lemma 5.4 applied with:

$$\begin{array}{lcl} n & \leftarrow & n-k \\ X & \leftarrow & \mathrm{Sh}_\tau\left(\mathrm{Tr}_\tau\left(\downarrow \Gamma_{reg}\right)\right) = \mathrm{Sh}_\tau\left(\mathrm{X}(\tau)\right) \\ P & \leftarrow & \mathrm{Sh}_\tau\left(P_\tau\right) \\ K & \leftarrow & \mathrm{Sh}_\tau\left(K_\tau\right) \end{array}$$

implies:

$$\mathrm{Sh}_\tau\left(\mathrm{Tr}_\tau\left(\downarrow \Gamma_{min}\right)\right) = \mathrm{Sh}_\tau\left(\mathrm{Tr}_\tau\left(\downarrow \Gamma_{reg}\right)\right)$$

In other words, Γ_{min} and Γ_{reg} coincide on simplices with bounding weight $\mu_B^{\neq}$, a contradiction with the definition of $\mu_B^{\neq}$. $\square$

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7 Appendices

A Invariance by global weight translation

Observation 7.1. *Let $\psi_\lambda : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$, be the transformation that shifts the weight by λ:*

$$\psi_\lambda(P, \mu) = (P, \mu + \lambda)$$

Let $\sigma \in K_{\mathbf{P}}$ be a simplex, from definitions 3.3 and 3.2 we have:

$$\begin{array}{lcl} P_C(\psi_\lambda(\sigma)) = P_C(\sigma) & \text{and} & \mu_C(\psi_\lambda(\sigma)) = \mu_C(\sigma) - \lambda \\ P_B(\psi_\lambda(\sigma)) = P_B(\sigma) & \text{and} & \mu_B(\psi_\lambda(\sigma)) = \mu_B(\sigma) - \lambda \end{array}$$

It follows that a global shift by a constant value λ results in an opposite shift on the weights of generalized circumcenters. It therefore preserves the relative order between simplices weights μ_C and μ_B:

$$\mu_C(\sigma_1) \leq \mu_C(\sigma_2) \iff \mu_C(\psi_\lambda(\sigma_1)) \leq \mu_C(\psi_\lambda(\sigma_2))$$

and the same relation holds for μ_B. Since the order $\leq$ between simplices (5) defined in section 3.3 relies entirely on comparisons on μ_C and μ_B, this total order is preserved by a global weight translation.

B Proof of Lemma 3.4

Proof. If $\sigma = \{(P_0, \mu_0), \ldots, (P_k, \mu_k)\}$, $|\sigma|$ is the convex hull of $\{P_0, \ldots, P_k\}$. If $P \notin |\sigma|$ the projection of P on $|\sigma|$ decreases the weighted distance from P to all the vertices of σ, which shows that (P, μ) cannot realize the arg min in (2). □

C Proof of Lemma 3.5

Figure 1 illustrates the possibilities for $\Theta(\sigma)$ in the case $n = 3$ and zero weights.

Proof. In (2), denote by $\tau \subset \sigma$ the set of vertex $(P_i, \mu_i) \in \sigma$ for which:

$$\mathrm{D}\left((P, \mu), (P_i, \mu_i)\right) = 0$$

This set cannot be empty as, if all inequalities in (2) where strict, a strictly smaller value of μ would still match the inequality, which would contradict with the arg min in (2). One has then $\tau \neq \varnothing$ and:

$$\forall (P_i, \mu_i) \in \tau, \mathrm{D}\left((P, \mu), (P_i, \mu_i)\right) = 0 \tag{33}$$

and of course:

$$\forall (P_i, \mu_i) \in \tau, \mathrm{D}\left((P, \mu), (P_i, \mu_i)\right) \leq 0 \tag{34}$$

No other (P, μ) with a smaller value of μ can satisfies (33) nor (34) as it would again similarly contradict the arg min in (2). It follows that:

$$(P_B, \mu_B)(\sigma) = (P_B, \mu_B)(\tau) = (P_C, \mu_C)(\tau)$$

□

D Proof of Proposition 4.3

Proof. Note that Γ_{reg} is unique under the assumed general position.

Since, in the regular triangulation, all $(n-1)$-simplices that are not on the boundary of $\mathcal{CH}(\mathbf{P})$ are shared by exactly two n-simplices, while only those in $\beta_{\mathbf{P}}$ have a single n-coface, we have:

$$\partial \Gamma_{reg} = \beta_{\mathbf{P}}$$

We claim now that :

$$\partial\Gamma = \beta_{\mathbf{P}} \Rightarrow \|\Gamma_{reg}\|_{(p)} \leq \|\Gamma\|_{(p)} \tag{35}$$

Indeed, (8) and (9) gives:

$$\|\Gamma\|_{(p)}^p = \sum_{\sigma\in\Gamma} \int_{|\sigma|} \delta_\sigma(x)^p dx = \int_{\mathcal{CH}(\mathbf{P})} \sum_{\substack{\sigma\in\Gamma \\ |\sigma|\ni x}} \delta_\sigma(x)^p dx$$

We get:

$$\|\Gamma\|_{(p)}^p = \int_{\mathcal{CH}(\mathbf{P})\setminus|K_{\mathbf{P}}^{n-1}|} \sum_{\substack{\sigma\in\Gamma \\ |\sigma|\ni x}} \delta_\sigma(x)^p dx \tag{36}$$

From the equivalence between regular triangulations and convex hull on lifted points we know that if $\sigma_{reg} \in \Gamma_{reg}$, then for any n-simplex in $\sigma \in K$:

$$x \in |\sigma| \cap |\sigma_{reg}| \Rightarrow \delta_{\sigma_{reg}}(x) \leq \delta_\sigma(x) \tag{37}$$

According to Lemma 7.2, in (36), there is an odd number of, and therefore at least one, simplex $\sigma \in \Gamma$ satisfying $|\sigma| \ni x$ in the condition on the sum. Therefore (37) gives:

$$x \in |\sigma_{reg}| \Rightarrow \delta_{\sigma_{reg}}(x)^p \leq \sum_{\substack{\sigma\in\Gamma \\ |\sigma|\ni x}} \delta_\sigma(x)^p \tag{38}$$

And, since, from definition of triangulation, for $x \in \mathcal{CH}(\mathbf{P}) \setminus |K_{\mathbf{P}}^{n-1}|$ there is exactly one simplex σ_{reg} such that $|\sigma_{reg}| \ni x$, (38) can be rewritten as:

$$\sum_{\substack{\sigma\in\Gamma_{reg} \\ |\sigma|\ni x}} \delta_\sigma(x)^p \leq \sum_{\substack{\sigma\in\Gamma \\ |\sigma|\ni x}} \delta_\sigma(x)^p \tag{39}$$

which, together with (36) gives the claim (35).

Now, if some n-simplex $\sigma \in \Gamma$ with $|\sigma| \ni x$, for some $x \in \mathcal{CH}(\mathbf{P}) \setminus |K_{\mathbf{P}}^{n-1}|$, is not Delaunay, then

$$\sum_{\substack{\sigma\in\Gamma_{reg} \\ |\sigma|\ni x}} \delta_\sigma(x)^p = \delta_{\sigma_{reg}}(x)^p < \sum_{|\sigma|\ni x} \delta_\sigma(x)^p$$

and since the function is continuous, this implies $\|\Gamma\|_{(p)} > \|\Gamma_{reg}\|_{(p)}$. □

Lemma 7.2. *Given $\mathbf{P} = \{P_1, \ldots, P_N\} \subset \mathbb{R}^n$, with $N \geq n+1$, in general position, denote by $\beta_{\mathbf{P}} \in C_{n-1}(K_{\mathbf{P}})$ the $(n-1)$-chain made of simplices belonging to the boundary of $\mathcal{CH}(\mathbf{P})$.*
Let $\Gamma \in C_n(K_{\mathbf{P}})$ be such that:

$$\partial\Gamma = \beta_{\mathbf{P}}$$

If $x \in \mathcal{CH}(\mathbf{P}) \setminus |K_{\mathbf{P}}^{n-1}|$ then there is an odd number of n-simplices $\sigma \in \Gamma$ such that $x \in |\sigma|$.

Proof. We claim that since $x \in \mathcal{CH}(\mathbf{P}) \setminus |K_{\mathbf{P}}^{n-1}|$ there is $x_\star \in \mathbb{R}^n \setminus \mathcal{CH}(\mathbf{P})$ such that:

$$[x_\star\, x] \cap |K_{\mathbf{P}}^{n-2}| = \varnothing \tag{40}$$

where $[x_\star\, x]$ denote the line segment in $\mathbb{R}^n$ between $x_\star$ and x.

Indeed, we consider moving a point x_t from x_0 to some x_1, picking x_0, x_1 far away enough to have $[x_0\, x_1] \cap \mathcal{CH}(\mathbf{P}) = \varnothing$ and in such a way that $[x_0\, x_1]$ belongs to none of the affine hyper-planes spanned by $(n-1)$-simplices in $K_{\mathbf{P}}$), which occurs generically. Then, the negation of condition (40) with $x_\star = x_t$, occurs only as isolated values of t. We can pick a value $t^\star$ for which is does not occur: set $x_\star = x_{t^\star}$ and the claim is proved. | Next, we navigate a point $y(t) = (1-t)x_\star + tx$ along segment $[x_\star\, x]$. This segment intersect transversally the $(n-1)$-faces $|\tau|$, for $\tau \in K_{\mathbf{P}}$. At each intersection point we can keep track of the change in the number of covering n-simplices, where, by *covering simplices* we name the n-simplices $\sigma \in \overline{\Gamma}$ such that $x \in |\sigma|$.

We know that this number is zero at $x_\star$ since $x_\star \notin \mathcal{CH}(\mathbf{P})$. Since $\mathcal{CH}(\mathbf{P}) \cap [x_\star\, x]$ is convex there is a single intersection point $y(t_b)$ between $[x_\star\, x]$ ans the boundary of $\mathcal{CH}(\mathbf{P})$.

This point $y(t_b) \in (x_\star, x)$ hits a $(n-1)$-simplex $\tau_b \in |\beta_{\mathbf{P}}|$, face of the convex hull boundary. Since $\partial\Gamma = \beta_{\mathbf{P}}$, we know that τ_b is shared by an odd number n_b of n-simplices in $|\Gamma|$. By definition of the convex hull, and since $\mathbf{P}$ is in general position, for each n-simplex σ coface of τ_b, $|\sigma|$ is on the inner side of the convex hull supporting half plane. It follows that the number of covering simplices become the odd number n_b just after the first crossing.

Then when crossing any other $(n-1)$-simplex $\tau_i \notin |\beta_{\mathbf{P}}|$, at some point $y(t_i)$, the condition $\partial\Gamma = \beta_{\mathbf{P}}$ requires the number n_i of n-simplices in $|\Gamma|$ coface of τ_i to be even. When crossing $|\tau_i|$, along $[x_\star\, x]$, point $y(t_i)$ exits k^-, and enters k^+ n-simplices in $|\Gamma|$, with $k^- + k^+ = n_i$. The current number of covering n-simplices value is incremented by $k^+ - k^-$. Since n_i is even and:

$$k^+ - k^- = k^+ + k^- - 2k^- = n_i - 2k^-$$

$k^+ - k^-$ is even and the number of covering simplices remains odd all along the path $[x_\star\, x]$. □

E Proof of Lemma 4.4

Proof. Since the expression (6) does not depend on the origin O, let us choose this origin to be $O = P_B(\sigma)$. With barycentric coordinates based on the vertices $P_i, i = 0, k$ of σ, i.e. $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$ such that $x = \sum_i \lambda_i P_i$, the expression of f_σ is:

$$f_\sigma(x) = \left(\sum_i \lambda_i \left((P_i - P_B(\sigma))^2 - \mu_i \right) \right) - (x - P_B(\sigma))^2$$

one has $(P_i - P_B(\sigma))^2 - \mu_i - \mu_B(\sigma) = \mathrm{D}\left((P_B(\sigma), \mu_B(\sigma)), (P_i, \mu_i)\right) \leq 0$ so that $(P_i - P_B(\sigma))^2 - \mu_i \leq \mu_B(\sigma)$ and it follows that:

$$\forall x, f_\sigma(x) \leq \mu_B(\sigma) - (x - P_B(\sigma))^2 \tag{41}$$

We have from Lemma 3.4 that $P_B(\sigma) \in |\Theta(\sigma)| \subset |\sigma|$ so that, in the expression of $P_B(\sigma)$ as a barycenter of vertices of σ, only coefficients λ_i corresponding to vertices of $\Theta(\sigma) \subset \sigma$ are non zero:

$$P_B(\sigma) = \sum_{(P_i, \mu_i) \in \Theta(\sigma)} \lambda_i P_i$$

One has by definition of (P_C, μ_C):

$$(P_i, \mu_i) \in \Theta(\sigma) \Rightarrow (P_i - P_C(\Theta(\sigma)))^2 - \mu_i = \mu_C(\Theta(\sigma))$$

and since we know from Lemma 3.5 that:

$$(P_B, \mu_B)(\sigma) = (P_B, \mu_B)(\Theta(\sigma)) = (P_C, \mu_C)(\Theta(\sigma))$$

one gets:

$$(P_i, \mu_i) \in \Theta(\sigma) \Rightarrow (P_i - P_B(\sigma))^2 - \mu_i = \mu_B(\sigma)$$

and:

$$f_\sigma(P_B(\sigma)) = \left(\sum_{(P_i, \mu_i) \in \Theta(\sigma)} \lambda_i((P_i - P_B(\sigma))^2 - \mu_i) \right) - 0^2 = \mu_B(\sigma)$$

This with (41) ends the proof. □

F Proof of Lemma 4.5

Proof. Consider the smallest ratio between two bounding of n-simplices in $K_{\mathbf{P}}$:

$$\iota = \inf_{\substack{\sigma_1, \sigma_2 \in K_{\mathbf{P}}^{[n]} \\ \mu_B(\sigma_1) < \mu_B(\sigma_2)}} \frac{\mu_B(\sigma_2)}{\mu_B(\sigma_1)}$$

From the finiteness of simplices, one has $\iota > 1$ and since for a finite number of n-simplices $\sigma \in K_{\mathbf{P}}^{[n]}$ we have from (11):

$$\lim_{p\to\infty} w_p(\sigma) = \mu_B(\sigma)$$

It follows that there is some p_0 large enough such that for any $p > p_0$ one has:

$$\forall \sigma_1, \sigma_2 \in K_{\mathbf{P}}^{[n]},\ \mu_B(\sigma_1) < \mu_B(\sigma_2) \Rightarrow \frac{\mu_B(\sigma_2)}{\mu_B(\sigma_1)} > \frac{1+\iota}{2}$$

Let $\mathcal{N}$ be total number of n-simplices in $K_{\mathbf{P}}$. Taking:

$$p^\star = \max\left(p_0, \left\lceil \frac{\log \mathcal{N}}{\log \frac{1+\iota}{2}} \right\rceil\right)$$

realizes the statement of the lemma. □

G Proof of Lemma 4.6

Proof of Lemma 4.6. Under generic condition, σ is in the regular triangulation of $\mathbf{P}$ if and only if there is a weighted point $(P_C(\sigma), \mu_C(\sigma))$ such that:

$$\forall (P_i, \mu_i) \in \sigma, \qquad \mathrm{D}\left((P_C(\sigma), \mu_C(\sigma)), (P_i, \mu_i)\right) = 0 \tag{42}$$
$$\forall (P_i, \mu_i) \in \mathbf{P} \setminus \sigma, \quad \mathrm{D}\left((P_C(\sigma), \mu_C(\sigma)), (P_i, \mu_i)\right) > 0 \tag{43}$$

Observe that, since $\tau \subset \sigma$, (42) implies:

$$\forall (P_i, \mu_i) \in \tau, \mathrm{D}\left((P_C(\sigma), 0), (P_i, \mu_i)\right) = \mu_C(\sigma)$$

This and the definition (12) of bisector show that $P_C(\sigma)$ must be on the bisector of τ:

$$P_C(\sigma) \in \mathbf{bis}_\tau \tag{44}$$

We get, for $(P_i, \mu_i) \in \mathbf{P}$:

$$\begin{aligned}
&\mathrm{D}\left((P_C(\sigma), \mu_C(\sigma)), (P_i, \mu_i)\right) \\
&\quad = (P_C(\sigma) - P_i)^2 - \mu_C(\sigma) - \mu_i \\
&\quad = (\pi_{\mathbf{bis}_\tau}(P_i) - P_C(\sigma))^2 + d(P_i, \mathbf{bis}_\tau)^2 - \mu_C(\sigma) - \mu_i \\
&\quad = \mathrm{D}_{\mathbf{bis}_\tau}\left((P_C(\sigma), \mu_C(\sigma)), (\pi_{\mathbf{bis}_\tau}(P_i), \mu_i - d(P_i, \mathbf{bis}_\tau)^2)\right) \\
&\quad = \mathrm{D}_{\mathbf{bis}_\tau}\left((P_C(\sigma), \mu_C(\sigma)), \pi_\tau(P_i, \mu_i)\right)
\end{aligned} \tag{45}$$

In the last two lines, the weighted distance is denoted $\mathrm{D}_{\mathbf{bis}_\tau}$ instead of D in order to stress that, thanks to (44) , it occurs on weighted point of $\mathbf{bis}_\tau \times \mathbb{R}$ rather than $\mathbb{R}^n \times \mathbb{R}$. It result that (42) and (43) are equivalent to:

$$\forall (P_i, \mu_i) \in \sigma, \qquad \mathrm{D}_{\mathbf{bis}_\tau}\left((P_C(\sigma), \mu_C(\sigma)), \pi_\tau(P_i, \mu_i)\right) = 0$$
$$\forall (P_i, \mu_i) \in \mathbf{P} \setminus \sigma, \quad \mathrm{D}_{\mathbf{bis}_\tau}\left((P_C(\sigma), \mu_C(\sigma)), \pi_\tau(P_i, \mu_i)\right) > 0$$

which precisely means that $\pi_\tau(\sigma)$ is a coface of the vertex $\pi_\tau(\tau) = \{(o_\tau, -\mu_C(\tau))\}$ in the regular triangulation of $\pi_\tau(\mathbf{P})$. □

H Proof of Lemma 4.13

Proof. 1. follows from Corollary 4.7 item 1. together with Proposition 4.2.

2. follows from Corollary 4.7 item 2. together with Proposition 4.2.

For 3. consider a simplex $\sigma \in \mathrm{X}(\tau)$. $\Phi_\tau(\sigma)$ is a simplex in the link of $\Phi_\tau(\tau)$ in the lower convex hull of $\Phi_\tau(\mathbf{P})$ and therefore the convex cone CC_σ with apex $\Phi_\tau(\tau)$ and going through $\Phi_\tau(|\sigma|)$ is on the boundary of the convex cone $CC_{\mathbf{P}}$ with apex $\Phi_\tau(\tau)$ and going through the convex hull of $\Phi_\tau(\mathbf{P})$.

Therefore $\mathrm{Sh}_\tau(\sigma)$, intersection of CC_σ with $\mathbf{bis}_\tau$ is on the boundary of the shadow polytope, intersection of $CC_\mathbf{P}$ with $\mathbf{bis}_\tau$.

$\mathrm{Sh}_\tau(\sigma)$ is bounded as being the convex hull of the shadow of its vertices. Thanks to observation 4.12, it is a facets of the boundary of the shadow polytope visible from 0. In the reverse direction, a bounded facet of the boundary of the shadow polytope is precisely the shadow of a simplex $\Phi_\tau(\sigma)$ in the link of $\Phi_\tau(\tau)$ in the lower convex hull of $\Phi_\tau(\mathbf{P})$ with all its vertices in P_τ. Therefore $\sigma \in \mathrm{X}(\tau)$. □

I Proof of Lemma 5.2

Proof. We need to prove that for any simplex σ in the link of τ, one has:

$$\partial \operatorname{Tr}_\tau(\Gamma)(\sigma) = \operatorname{Tr}_\tau(\partial\Gamma)(\sigma)$$

We have by definition that a $(n-k-2)$-simplex σ is in $\operatorname{Tr}_\tau(\partial\Gamma)(\sigma)$ if and only if $\tau \cap \sigma = \varnothing$ and $\tau \cup \sigma$ is in $\partial\Gamma$. In other words, $\tau \cup \sigma$ has an odd number of n-cofaces in the chain Γ. This in turn means that σ has an odd number of $(n-k)$-cofaces in the trace of Γ in the link of τ, i.e. $\sigma \in \partial \operatorname{Tr}_\tau(\Gamma)$. □

J Proof of Lemma 5.3

Proof. Assume for a contradiction that x is in the relative interior of X, that is there is some $\rho > 0$ such that $B(x,\rho) \cap X = B(x,\rho) \cap \partial C$. Then all facets containing x are visible from O. if x is not a vertex pf C, it belongs then to the relative interior of a convex face f in ∂C with $\dim f \geq 1$. Then we have a contradiction since the function $y \mapsto d(P, y)$ is convex on f and cannot have an interior local maximum at x. We assume now that x is a vertex of C.

Following for example [9, 10], denote by $\operatorname{Tan}_x C$ and $\operatorname{Nor}_x C$ respectively the Tangent and Normal cone to C at x. In case of a closed polytope they can be expressed as:

$$\operatorname{Tan}_x C = \bigcap_{\rho > 0} \{\lambda(c - x), \lambda \geq 0, c \in \mathcal{B}(x,\rho)\}$$

and

$$\operatorname{Nor}_x C = (\operatorname{Tan}_x C)^\perp \underset{\text{def.}}{=} \{u, \forall v \in \operatorname{Tan}_x C, \langle u, v\rangle \leq 0\} \tag{46}$$

Since C is a convex polytope, $\operatorname{Tan}_x C$ is a convex closed cone and one has [10]:

$$\operatorname{Tan}_x C = (\operatorname{Nor}_x C)^\perp \underset{\text{def.}}{=} \{v, \forall u \in \operatorname{Nor}_x C, \langle u, v\rangle \leq 0\} \tag{47}$$

Each facet F_i of C containing x is supported by a half-space $H_i = \{y, \langle y - x, n_i\rangle \leq 0\}$ and one has:

$$\operatorname{Nor}_x C = \left\{\sum_i \lambda_i n_i, \forall i, \lambda_i \geq 0\right\} \tag{48}$$

$$\forall i, \langle O - x, n_i\rangle > 0$$

This with (48) gives:

$$\forall u \in \operatorname{Nor}_x C, \langle O - x, u\rangle > 0 \tag{49}$$

using (47) we get that $O - x$ is in the interior of $-\operatorname{Tan}_x C$ i.e.:

$$x - O \in (\operatorname{Tan}_x C)^\circ \tag{50}$$

Since x is a vertex of C, $\operatorname{Tan}_x C$ is the convex hull of $\operatorname{Tan}_x \partial C$, therefore (50) implies at there are $t_1, \ldots t_{n+1} \in \operatorname{Tan}_x \partial C$ and $\lambda_1, \ldots \lambda_{n+1} \geq 0$ such that:

$$x - O = \sum_i \lambda_i t_i$$

which gives:

$$0 < \langle x - O, x - O \rangle = \langle x - O, \sum_i \lambda_i t_i \rangle = \sum_i \lambda_i \langle x - O, t_i \rangle$$

Since all the λ_i are not negative, there must be at least one i for which:

$$\langle x - O, t_i \rangle > 0$$

This precisely means that $y \mapsto d(P, y)$ is increasing in the direction t_i in a neighborhood of x in X, a contradiction since x is assumed to be a local maximum of $y \mapsto d(P, y)$ in X. □

K Proof of Lemma 6.1

Proof. Using the definition 3.3, one has:

$$(P_C, \mu_C)\,(\pi_\tau(\tau) \cup \pi_\tau(\sigma)) = \underset{\substack{(P,\mu)\in\mathbb{R}^n\times\mathbb{R} \\ \forall (P_i,\mu_i)\in\pi_\tau(\tau)\cup\pi_\tau(\sigma), \mathrm{D}\left((P,\mu),(P_i,\mu_i)\right)=0}}{\arg\min} \mu \tag{51}$$

Looking at Definition 3.3 in the light of (45) in the proof of Lemma 4.6 in Appendix G, we get that:

$$\sigma \supset \tau \;\Rightarrow\; \mu_C(\sigma) = \mu_C(\pi_\tau(\sigma)) \text{ and } P_C(\sigma) = P_C(\pi_\tau(\sigma)) \tag{52}$$

Since both terms of (28) are invariant by a global translation, we can assume without loss of generality and in order to make the computations simpler that $o_\tau = 0$.

In this case, as seen in (14) the coordinates of $\pi_\tau(\tau)$ are $(0, -\mu_C(\pi_\tau(\tau)) = (0, -\mu_C(\tau))$ by (52). So that $\mathrm{D}\left((P, \mu), \pi_\tau(\tau)\right) = 0$ gives us:

$$\mu = P^2 + \mu_C(\tau) \tag{53}$$

It follows that among the weighted points (P, μ) that satisfy $\mathrm{D}\left((P, \mu), \pi_\tau(\tau)\right) = 0$, minimizing μ is equivalent to minimizing P^2 and one can reformulate the characterization (51) of $(P_C, \mu_C)\,(\pi_\tau(\tau \cup \sigma)) = (P_C, \mu_C)\,(\pi_\tau(\tau) \cup \pi_\tau(\sigma))$ as:

$$(P_C, \mu_C)\,(\pi_\tau(\tau) \cup \pi_\tau(\sigma)) = \underset{\substack{(P,\mu)\in\mathbb{R}^n\times\mathbb{R} \\ \forall (P_i,\mu_i)\in\pi_\tau(\tau)\cup\pi_\tau(\sigma), \mathrm{D}\left((P,\mu),(P_i,\mu_i)\right)=0}}{\arg\min} P^2 \tag{54}$$

For $(P, \mu) \in \mathbb{R}^{n-k} \times \mathbb{R}$, define the hyperplane $\Pi_{(P,\mu)}$ in $\mathbb{R}^{n-k} \times \mathbb{R}$ as:

$$(X, z) \in \Pi_{(P,\mu)} \underset{\text{def.}}{\iff} z = (\mu - P^2) + 2\langle P, X \rangle \tag{55}$$

Observe that:

$$\mathrm{D}\left((P, \mu), (P_i, \mu_i)\right) = 0 \iff \mathrm{lift}(P_i, \mu_i) \in \Pi_{(P,\mu)}$$

So that the definition of $(P_C, \mu_C)\,(\pi_\tau(\tau) \cup \pi_\tau(\sigma))$ given in (54) can be equivalently formulated as $\Pi_{(P_C,\mu_C)}$ being the hyperplane in $\mathbb{R}^{n-k} \times \mathbb{R}$ that minimizes P_C^2 among all hyperplanes containing both $\mathrm{lift}(\pi_\tau(\tau))$ and all the points in $\mathrm{lift}(\pi_\tau(\sigma))$.

But, as seen on (55), $2\|P_C\|$ is the slope of the hyperplane $\Pi_{(P_C,\mu_C)}$ so that $\Pi_{(P,\mu)}$ is the hyperplane with minimal slope going through $\Phi_\tau(\tau) = \mathrm{lift}(\pi_\tau(\tau))$ and all the points in $\Phi_\tau(\sigma) = \mathrm{lift}(\pi_\tau(\sigma))$. This slope $2\|P_C\|$ is also the slope of the unique $(\dim(\sigma)+1)$-dimensional affine space F going through $\Phi_\tau(\tau) = \mathrm{lift}(\pi_\tau(\tau)) = (0, \mu_C(\tau))$ and all the points in $\Phi_\tau(\sigma) = \mathrm{lift}(\pi_\tau(\sigma))$. Since $F \cap \mathbb{R}^{n-k} \times \{0\}$ is the affine space supporting $\mathrm{Sh}_\tau(\sigma)$, one has:

$$d_0(\mathrm{Sh}_\tau(\sigma)) = \frac{\mu_C(\tau)}{2\|P_C\,(\pi_\tau(\tau) \cup \pi_\tau(\sigma))\,\|}$$

so that, using (52) for the second equality:

$$d_0(\mathrm{Sh}_\tau(\sigma)) = \frac{\mu_C(\tau)}{2\sqrt{\mu_C\left(\pi_\tau(\tau)\cup\pi_\tau(\sigma)\right)-\mu_C(\tau)}} = \frac{\mu_C(\tau)}{2\sqrt{\mu_C\left(\tau\cup\sigma\right)-\mu_C(\tau)}}$$

It follows that the map:

$$\mu_C(\tau\cup\sigma) \to d_0(\mathrm{Sh}_\tau(\sigma))$$

is decreasing. □

L Proof of Lemma 6.2

Proof. If we denote by $\underset{\rho}{\to}\Gamma$ the set of simplex in Γ with bounding weight equal to ρ:

$$\underset{\rho}{\to}\Gamma \underset{\text{def.}}{=} \{\sigma\in\Gamma, \mu_B(\sigma)=\rho\}$$

We claim that:

$$(\downarrow_\rho \Gamma_1 \sqsubseteq_{lex} \downarrow_\rho \Gamma_2) \;\Rightarrow\; \left(\underset{\rho}{\to}\Gamma_1 \sqsubseteq_{lex} \underset{\rho}{\to}\Gamma_2\right) \tag{56}$$

Indeed, by definition of the lexicographic order, if this did not holds, it would imply $\underset{\rho}{\to}\Gamma_1 \neq \underset{\rho}{\to}\Gamma_2$ and the largest simplex for which $\underset{\rho}{\to}\Gamma_1$ and $\underset{\rho}{\to}\Gamma_2$ differ would be in Γ_1 contradicting $\downarrow_\rho \Gamma_1 \sqsubseteq_{lex}\downarrow_\rho \Gamma_2$ which proves the claim (56).

Note that, from Lemma 3.5 and generic condition 3.2, all the simplices in $\underset{\rho}{\to}\Gamma_1$ and $\underset{\rho}{\to}\Gamma_2$ are in the star of a single simplex $\tau=\tau_\rho$ such that $\mu_C(\tau)=\mu_B(\tau)=\rho$.

It remains to show that the order $\sqsubseteq_{lex}$ restricted to simplices $\tau\cup\sigma$ with $\mu_B(\tau\cup\sigma)=\rho$ corresponds to the order $\sqsubseteq_{\mathrm{Sh}}$ on the shadow of σ.

By definition of $\sqsubseteq_{lex}$, since in (5) one has always $\mu_0(\tau\cup\sigma_1)=\mu_0(\tau\cup\sigma_2)=\rho$, it goes like this:

$$\sigma_1<\sigma_2 \underset{\text{def.}}{\iff} \begin{cases} \exists k\geq 1,\ \mu_k(\tau\cup\sigma_1)>\mu_k(\tau\cup\sigma_2) \\ \text{and } \forall j, 0\leq j<k,\ \mu_j(\tau\cup\sigma_1)=\mu_j(\tau\cup\sigma_2)\end{cases} \tag{57}$$

Observe that this expression is similar to (21).

For a 0-simplex $\{v\}\in K_{\mathbf{P}}$, the circumweight $\mu_C(\tau\cup\{v\})$ is, according to Lemma 6.1, a decreasing function of the distance $d_0(\mathrm{Sh}_\tau(\eta))$ between its shadow and the origin. It follows that for a $(n-k-1)$-simplex $\sigma\in K_{\mathbf{P}}$, the vertex v for which the circumweight $\mu_C(\tau\cup\{v\})$ is minimal has its shadow $\mathrm{Sh}_\tau(v)$ maximizing the distance to the origin. This minimal circumweight is $\mu_1(\tau\cup\{v\})$ while this maximal distance is $\delta_0(\sigma)$.

More generally, looking at (20) and (4), Lemma 6.1 allows to check that the simplex $\Theta_k(\sigma)$ of (4) in the star of τ in $K_{\mathbf{P}}$ corresponds to the simplex $\tau_{k-1}(\sigma)$ in (20) in the link of τ:

$$\Theta_k(\sigma)=\tau\cup\tau_{k-1}(\sigma)$$

So that for $\sigma_1,\sigma_2\in K_{\mathbf{P}}$ referring to (21) and (57):

$$\mu_k(\tau\cup\sigma_1)\leq\mu_k(\tau\cup\sigma_2) \iff \delta_{k-1}(\sigma_1)\geq\delta_{k-1}(\sigma_2)$$

It follows that, for $\Gamma_1,\Gamma_2\in \boldsymbol{C_n}(K_{\mathbf{P}})$ and $\tau=\tau_\rho$:

$$\underset{\rho}{\to}\Gamma_1 \sqsubseteq_{lex} \underset{\rho}{\to}\Gamma_2 \iff \begin{cases}\mathrm{Sh}_\tau\left(\mathrm{Tr}_{\tau_\rho}\left(\downarrow_\rho\Gamma_1\right)\right) \\ \sqsubseteq_{\mathrm{Sh}} \mathrm{Sh}_{\tau_\rho}\left(\mathrm{Tr}_{\tau_\rho}\left(\downarrow_\rho\Gamma_2\right)\right)\end{cases}$$

which, with claim (56), ends the proof. □